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One-Dimensional Boltzmann Equation with a Three-Body Collision Term

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It is shown that a linearized one-dimensional Boltzmann equation with a certain simple three-body collision term is trivially soluable.

KEY WORDS: Collision operator; eigenvalues.

1. INTRODUCTION

In this short note we present the solution to a one-dimensional linearized Boltzmann equation with a three-body collision term. A simple form of the collision term is assumed and the solution is quite trivial.

This note should serve some pedagogical purpose. The textbook example of a soluable Boltzmann equation has been the quasi-Maxwell model.[1,2] The one-dimensional model here is much simpler.

The simplicity of our model equation should allow the testing of many ideas in nonequilibrium statistical mechanics. Thus, this note should also serve some useful purpose for research.

It is surprising that there has not been extensive work on the kinetic theory of one-dimensional gas. An obvious reason might be that two-body collisions in one dimension are like no collisions at all (assuming equal mass for all particles). Thus, the one-dimensional gas looks uninteresting unless internal states of the particles are introduced or multiparticle collisions are considered. This note shows that three-body collisions do not necessarily make the problem complicated. According to our experience, internal states of the particles would make the problem intractable. We do emphasize that we have not derived the three-body collision terms from an

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interparticle potential energy. We have simply assumed a simple form of the collision terms. We have not figured out what potentials would produce such collision terms.

Another reason for the unpopularity of one-dimensional gas might be the lack of empirical realizations. However, recently there has been interest in plasmas in a very strong magnetic field. The motion of electrons along a field line can be approximately studied as a one-dimensional system.[3] Such systems will undoubtedly make the study of one-dimensional models more attractive in the future.

In this note we simply write down the Boltzmann equation, define the collision operator and give its eigenvalues and eigenvectors. Then we give one trivial application. There are many short applications such as the study of sound waves which the reader can easily analyze for pedagogical purposes.

2. THE BOLTZMANN EQUATION

2.1. The Collision Term

Consider a one-dimensional gas of identical particles of unit mass. We begin with the Boltzmann equation

$$\frac{\partial f}{\partial t} + p\frac{\partial f}{\partial x} + F\frac{\partial f}{\partial p} = \left(\frac{\partial f}{\partial t}\right)_c \tag{2.1}$$

$$\left(\frac{\partial f}{\partial t}\right)_c = \int dp'\, dp''\, dq\, dq'\, dq''\, R(p, p', p'', q, q', q'') \times \left[-f(p)f(p')f(p'') + f(q)f(q')f(q'')\right] \tag{2.2}$$

$$R(p, p', p'', q, q', q'') = \alpha\delta(p + p' + p'' - q - q' - q'') \times \delta\left[\tfrac{1}{2}(p^2 + p'^2 + p''^2) - \tfrac{1}{2}(q^2 + q'^2 + q''^2)\right] \tag{2.3}$$

In these equations, f is the one-particle distribution function. It is a function of position x, momentum p, and time t. In (2.1), F is the external force on a particle. In (2.2), all f have the same x and t. (2.2) is the three-body collision term. R is the rate of collisions $p + p' + p'' \leftrightarrow q + q' + q''$. As (2.3) shows, a collision must satisfy momentum and energy conservation. Now we make the simplifying assumption that α is a constant independent of p, p', p'', q, q', and q''. The constant α has the dimension of $1/(\text{density})^2(\text{time})$, i.e., a rate per density squared.

The assumption that α is a constant is crucial for the simplicity of the analysis below. As was mentioned above, we have not linked such a collision rate to any interaction energy. If the collision rate is derivable from quantum mechanics, then α is proportional to the T-matrix element

squared. Constant α means that intermediate-state energies are all much higher than the kinetic energies involved. It is certainly conceivable that there are other interactions which would satisfy the constant-α requirement approximately for energies of the order of the temperature.

2.2. Center-of-Mass Variables

Let P, E be the total three-body momentum and energy, respectively,

$$P = p + p' + p'' \tag{2.4}$$

$$2E = p^2 + p'^2 + p''^2 \tag{2.5}$$

Now imagine a fictitious three-dimensional space of vectors $\mathbf{p} = (p, p', p'')$. In this space, (2.4) defines a plane and (2.5) defines a spherical surface. This surface and the plane have an intersection, which is a circle of radius

$$r = \left[2E - 3(P/3)^2\right]^{1/2} = (2E - P^2/3)^{1/2} \tag{2.6}$$

Note that $r^2/2$ is the energy in the center-of-mass frame, $P/\sqrt{3}$ is the distance from the plane to the origin.

Let

$$\begin{aligned} Q &= q + q' + q'' \\ s &= \tfrac{1}{2}\left(q^2 + q'^2 + q''^2 - \tfrac{1}{3}\,Q^2\right) \end{aligned} \tag{2.7}$$

then the energy momentum conservation demands that $\mathbf{q} = (q, q', q'')$ be on the same circle defined above by $\mathbf{p}$, i.e., $P = Q$ and $r = s$. Let ϕ be the angle measured along the circle. Then we have

$$\begin{aligned} dq\,dq'\,dq''\,R(p, p', p'', q, q', q'') &= d\frac{Q}{\sqrt{3}}\,r\,dr\,d\phi\,\delta(P - Q)\delta\left(\frac{r^2}{2} - \frac{s^2}{2}\right)\alpha \\ &= \frac{\alpha}{\sqrt{3}}\,d\phi \end{aligned} \tag{2.8}$$

The q's can therefore be expressed in terms of ϕ, P, and r. Draw a line from the center of the circle to the intersection of the plane and p axis. Choose this line as $\phi = 0$. Then, a little algebra gives

$$\begin{aligned} q &= \frac{P}{3} + r\sqrt{\frac{2}{3}}\cos\phi \\ q' &= \frac{P}{3} - r\frac{1}{\sqrt{6}}\cos\phi - r\frac{1}{\sqrt{2}}\sin\phi \\ q'' &= \frac{P}{3} - r\frac{1}{\sqrt{6}}\cos\phi + r\frac{1}{\sqrt{2}}\sin\phi \end{aligned} \tag{2.9}$$

We have set $Q = P$, $s = r$.

2.3. Collision Rate

The collision rate can be defined as

$$\nu = \int dp'\, dp''\, dq\, dq'\, dq''\, f(p')f(p'')R(p, p', p'', q, q', q'') \tag{2.10}$$

In view of (2.8), we have

$$\nu = \frac{2\pi}{\sqrt{3}}\alpha n^2 \tag{2.11}$$

$$n = \int dp\, f(p) \tag{2.12}$$

n is the particle density.

Note that ν is independent of p, i.e., the rate is independent of velocity of the particle, but only on the density n. It sets a temperature independent time scale at a fixed n.

3. THE COLLISION OPERATOR

3.1. The Linearized Equation

Write

$$\begin{aligned} f_0(p) &= n(2\pi T)^{-1/2} e^{-p^2/2T} \\ f(x, p, t) &= f_0(p)\left[1 + \varphi(x, p, t)\right] \end{aligned} \tag{3.1}$$

in the Boltzmann equation (2.1) and (2.2), and keep only first-order terms in φ, we obtain the linearized equation

$$\frac{\partial \varphi}{\partial t} + p\frac{\partial \varphi}{\partial x} + F\frac{\partial \varphi}{\partial p} = -K\varphi \tag{3.2}$$

where K is the collision operator defined by

$$\begin{aligned} K\varphi = \int dp'\, dp''\, dq\, dq'\, dq''\, f_0(p')f_0(p'')R \\ \times\left[\varphi(p) + \varphi(p') + \varphi(p'') - \varphi(q) - \varphi(q') - \varphi(q'')\right] \end{aligned} \tag{3.3}$$

All the φ's in (3.3) have the same (x, t). Note that

$$f_0(p)f_0(p')f_0(p'') = f_0(q)f_0(q')f_0(q'') \tag{3.4}$$

under the energy conservation condition demanded by R. By the symmetry between p' and p'' and that among q, q', and q'', (3.3) is simplified to

$$\begin{aligned} K\varphi = \nu\left[\varphi(p) + \frac{2}{n}\int dp'\, f_0(p')\varphi(p')\right] \\ - 3\int dp'\, dp''\, dq\, dq'\, dq''\, f_0(p')f_0(p'')R\varphi(q) \end{aligned} \tag{3.5}$$

where some of the integrals are done by (2.8). We now proceed to obtain the eigenvalues and eigenvectors of K.

3.2. Spectrum of the Collision Operator

We define the linear vector space as functions of p with the scalar product $\langle\psi\varphi\rangle$ given by the average of $\psi(p)\varphi(p)$ over the Maxwell distribution $f_0(p)$. To save writing, let us set the temperature T and density n to unity, so that

$$\langle\psi\varphi\rangle = \int \frac{dp}{(2\pi)^{1/2}} e^{-p^2/2}\psi(p)\varphi(p) \tag{3.6}$$

We choose a set of orthonormal functions φ_n as a basis for the space:

$$\varphi_n = (2^n n!)^{-1/2} H_n(p/\sqrt{2}) \tag{3.7}$$

$$\langle\varphi_m\varphi_n\rangle = \delta_{mn} \tag{3.8}$$

H_n is the Hermite polynomial. We now proceed to show that these basis functions turn out to be the eigenvectors of K.

To calculate the matrix elements of K over the basis (3.7), it is convenient to define the generating function

$$g(\lambda, p) = \sum_{n=0}^{\infty} \varphi_n(p) \frac{(i\lambda)^n}{(n!)^{1/2}}$$

$$= e^{i\lambda p + \lambda^2/2} \tag{3.9}$$

The last equality is a consequence of the identity[4]

$$\sum_{n=0}^{\infty} H_n(x) \frac{z^n}{n!} = e^{2xz - z^2} \tag{3.10}$$

The quantity $\langle g(\lambda')Kg(\lambda)\rangle$ is easily calculated, and matrix elements can be obtained by expanding in powers of λ and λ'. (K_{mn} is the coefficient of $-\lambda^n\lambda'^m/(n!\,m!)^{1/2}$ of $\langle g(\lambda')Kg(\lambda)\rangle$.) From (3.5), we obtain

$$\langle g(\lambda')Kg(\lambda)\rangle = \nu\left[\langle g(\lambda')g(\lambda)\rangle + 2\langle g(\lambda')\rangle\langle g(\lambda)\rangle\right]$$
$$- 3\int dp\,dp'\,dp''\,dq\,dq'\,dq''\, f_0(p)f_0(p')f_0(p'')$$
$$\times Rg(\lambda', p)\, g(\lambda, q) \tag{3.11}$$

Note that

$$\langle g(\lambda)\rangle = \langle e^{i\lambda p}\rangle e^{\lambda^2/2} = 1$$
$$\langle g(\lambda')\, g(\lambda)\rangle = e^{-\lambda\lambda'} \tag{3.12}$$

Only the last integral of (3.11) is nontrivial. This integral can be evaluated with the help of the P, r, ϕ variables introduced in Section 2. We note that

$$\begin{aligned} f_0(p)f_0(p')f_0(p'') &= (2\pi)^{-3/2}\exp\left[-\tfrac{1}{2}(r^2 + P^2/3)\right] \\ dp\,dp'\,dp''\,dq\,dq'\,dq''\,R &= \frac{\alpha}{\sqrt{3}}\,d\psi\,d\phi\,\frac{P}{\sqrt{3}}\,r\,dr \\ p &= \frac{P}{3} + \sqrt{\frac{2}{3}}\,r\cos\psi \\ q &= \frac{P}{3} + \sqrt{\frac{2}{3}}\,r\cos\phi \end{aligned} \tag{3.13}$$

We need the help of one more identity,[5].

$$\begin{aligned} &\int_0^\infty J_0(\alpha t)J_0(\beta t)\exp(-\gamma^2 t^2)\,t\,dt \\ &\quad = \frac{1}{2}\gamma^{-2}\exp\left[-\tfrac{1}{4}\gamma^{-2}(\alpha^2 + \beta^2)\right]I_0\left(\tfrac{1}{2}\alpha\beta\gamma^{-2}\right) \end{aligned} \tag{3.14}$$

J_0, I_0 are Bessel functions. The ψ, ϕ integrals give J_0's and the r integral gives I_0. The P integral is trivial. We finally obtain

$$\begin{aligned} \langle g(\lambda')Kg(\lambda)\rangle &= \nu\left[e^{-\lambda\lambda'} + 2 - 3\exp\left(-\tfrac{1}{3}\lambda\lambda'\right)I_0\left(\tfrac{2}{3}\lambda\lambda'\right)\right] \\ I_0\left(\tfrac{2}{3}\lambda\lambda'\right) &= \sum_{m=0}^{\infty}\left(\tfrac{1}{3}\lambda\lambda'\right)^{2m}\frac{1}{(m!)^2} \end{aligned} \tag{3.15}$$

Since this is a function of $\lambda\lambda'$, all terms have equal power in λ and λ', i.e., $\langle\varphi_n K\varphi_m\rangle$ must be diagonal:

$$\langle\varphi_n K\varphi_m\rangle = \gamma_n\delta_{nm} \tag{3.16}$$

$$\gamma_n = \nu\left\{1 - 3\left[\frac{d^n}{dx^n}e^{(1/3)x}I_0\left(-\tfrac{2}{3}x\right)\right]_{x=0}\right\}, \qquad n = 1, 2, 3, \ldots \tag{3.17}$$

γ_n are the eigenvalues of K.

One easily verifies that $\gamma_0 = \gamma_1 = \gamma_2 = 0$. This is a consequence of the conservation of particle number, momentum, and energy. γ_n increases with n and approaches ν as $n \to \infty$. The lowest two nonzero eigenvalues are

$$\begin{aligned} \gamma_3 &= \tfrac{4}{9}\nu \\ \gamma_4 &= \tfrac{7}{9}\nu \end{aligned} \tag{3.18}$$

For very large n, one obtains

$$\gamma_n \to \nu\left[1 - 3\left(\frac{3}{4\pi n}\right)^{1/2}\right] \tag{3.19}$$

The first few eigenvectors are

$$\begin{aligned} \varphi_0 &= 1 \\ \varphi_1 &= p \\ \varphi_2 &= \frac{1}{\sqrt{2}}(p^2 - 1) \\ \varphi_3 &= \frac{1}{\sqrt{6}} p(p^2 - 3) \end{aligned} \tag{3.20}$$

Each p should be replaced by $p/\sqrt{T}$ if factors of T are explicitly written out.

4. HEAT CONDUCTION

As an example of applying the above results, let us study the heat conduction. Let us assume a steady state situation, i.e., f is time independent, and write

$$f(x, p) = f_0[T(x), n(x), p][1 + \varphi(x, p)] \tag{4.1}$$

and assume φ is small, proportional to the gradient of T and n. Here f_0 is the equilibrium distribution at the local temperature $T(x)$ and local density $n(x)$. φ describes the deviation from local equilibrium. The linearized Boltzmann equation becomes

$$p\left(\frac{\partial}{\partial T} \ln f_0\right) \frac{\partial T}{\partial x} + \frac{p}{n} \frac{\partial n}{\partial x} = -K\varphi \tag{4.2}$$

Since we have assumed that φ is proportional to the gradients, $\partial\varphi/\partial x$ is proportional to the square of the gradients of T and n. Equation (4.2) keeps only the first order in the gradients. External forces are excluded.

The temperature and the density are not independent. Multiplying (4.2) by pf_0 and integrating over p, one obtains

$$n \frac{\partial T}{\partial x} + T \frac{\partial n}{\partial x} = 0 \tag{4.3}$$

which simply says that the pressure nT must be uniform. Consequently, (4.2) becomes

$$\frac{\partial T}{\partial x} p(p^2 - 3T) \frac{1}{2T^2} = -K\varphi \tag{4.4}$$

Since $p(p^2 - 3T) = \sqrt{6}\, T^{3/2}\varphi_3$ [see Eq. (3.20)] K can be replaced by $\gamma_3 = \frac{4}{9}\nu$, then

$$\varphi = -\frac{9}{8\nu T^2} p(p^2 - 3T) \frac{\partial T}{\partial x} \tag{4.5}$$

This is the fractional deviation of f from local equilibrium. This deviation produces no mass current, but it produces an energy current

$$J = n\langle \varphi p^2/2 \times p \rangle$$
$$= -n \frac{27}{8\nu} \frac{\partial T}{\partial x} T \tag{4.6}$$

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REFERENCES

1. G. E. Uhlenbeck and G. W. Ford, *Lectures in Statistical Mechanics* (American Mathematical Society, Providence, Rhode Island, 1963), p. 85.
2. G. H. Wannier, *Statistical Physics* (Wiley, New York, 1966), p. 404.
3. T. M. O'Neil, preprint.
4. A. Erdelyi, ed., *Higher Transcendental Functions*, Vol. II (McGraw-Hill, New York, 1953), p. 194.
5. See p. 50 of Ref. 4.